Pure Literal Rules and QU-resolution in QBF Search-Based Solvers

Allen Van Gelder
Computer Science Dept.
Univ. of California
Santa Cruz, CA, USA

http://www.cse.ucsc.edu/∼avg/

http://www.cse.ucsc.edu/∼avg/Pos12/
These slides are pos12-trans.pdf

http://www.cse.ucsc.edu/∼avg/ProofChecker/
Software directory, contains QdpllexpSimple.tar.

What are Quantified Boolean Formulas (QBFs)?

Most general definition:

- Add quantification ($\forall u$, $\exists e$) as a new propositional operation.
- A quantified variable must be *true* or *false*.

Least general definition:

- $\mathcal{F}$ is a quantifier-free propositional formula in *conjunctive normal form*.
- $\overrightarrow{Q}$ is a sequence of quantified variables, outer to inner scopes.
- $\Phi = \overrightarrow{Q} \, . \, \mathcal{F}$ is a ***prefix CNF*** (PCNF) formula.
- This paper considers *closed* PCNF (all variables quantified).
- Example (chart form):

Φ	$\forall u$	$\forall v$	$\exists e$	$\exists f$	$\forall t$	$\exists d$
C_1	u			f	t	$\overline{d}$
C_2	$\overline{u}$	$\overline{v}$	$\overline{e}$		$\overline{t}$	$\overline{d}$
C_3		v	e	f	t	d
$\dots$						

Literal Naming Convention

- Lowercase letters near the beginning of the alphabet are *existential* literals (or variables, if specified in the context), e.g., c, d, e, etc.
- Lowercase letters near the end of the alphabet are *universal* literals (or variables, if specified in the context), e.g., t, u, v, etc.
- p, q, r, s are of unspecified quantifier type.
- $|p|$ denotes the *variable* underlying the *literal* p. (Mainly for quantifier sequences.)

QBF as a Two-Player Game

Two players: A is *Universal*, E is *Existential*.

$\Phi = \overrightarrow{Q}.\,\mathcal{F}$ is a PCNF (prefix: $\overrightarrow{Q}$, matrix: $\mathcal{F}$).

When outermost (unassigned) variable is *universal*, A chooses a value for it and Φ gets simplified.

When outermost (unassigned) variable is *existential*, E chooses a value for it and Φ gets simplified.

If Φ simplifies to *false*, A wins. If Φ simplifies to *true*, E wins.

"Definition": ***Truth-Value Semantics*** of Φ (coarse grain):

- The value of Φ is *false* (or 0) if and only if A has a winning strategy.
- The value of Φ is *true* (or 1) if and only if E has a winning strategy.

Example: Two-Player Game

	qblock		qblock		qblock	qblock
	← 1 →		← 2 →		3	4
Φ	$\forall u$	$\forall v$	$\exists e$	$\exists f$	$\forall t$	$\exists d$
C_1	u			f	t	$\overline{d}$
C_2	$\overline{u}$	$\overline{v}$	$\overline{e}$		$\overline{t}$	$\overline{d}$
C_3		v	e	f	t	d
...						

Say A chooses $u = 0$;
C_2 is "satisfied" and
is deleted from $\mathcal{F}$.
u is deleted from C_1.

Then say A chooses $v = 0$;
v is deleted from C_3.

Φ_1	$\exists e$	$\exists f$	$\forall t$	$\exists d$
C_1		f	t	$\overline{d}$
C_3	e	f	t	d
...				

Next, Player E chooses values
for e and f.

And so on.

Notes:

- Quantifier order matters between qblocks, but not within a qblock.
- *One* empty clause makes Φ false (the goal of A).
- *Every* clause must be satisfied to make Φ true (the goal of E), but *one* true literal suffices to satisfy a clause.

Tree Models for QBF

For a QBF $\Phi = \overrightarrow{Q} \cdot \mathcal{F}$, a ***tree model*** of Φ is a nonempty set of ***ordered assignments*** with certain restrictions that ensure that the set defines a tree.

An ***ordered assignment*** is a total assignment with the literals in quantifier-prefix order, outer to inner.

Each ordered assignment is a tree branch and satisfies all clauses.

Φ	$\forall u$	$\forall v$	$\exists e$	$\exists f$	$\forall t$	$\exists d$
C_1	u			f	t	$\overline{d}$
C_2	$\overline{u}$	$\overline{v}$	$\overline{e}$		$\overline{t}$	$\overline{d}$
C_3		v	e	f	t	d
	$\dots$					
	$\overline{u}$	$\overline{v}$	e	f	$\overline{t}$	$\overline{d}$
	$\overline{u}$	$\overline{v}$	e	f	t	d
	$\overline{u}$	v	e	f	$\overline{t}$	$\overline{d}$
	$\overline{u}$	v	e	f	t	d
	u	$\overline{v}$	e	f	$\overline{t}$	$\overline{d}$
	u	$\overline{v}$	e	f	t	d
	u	v	$\overline{e}$	f	$\overline{t}$	$\overline{d}$
	u	v	$\overline{e}$	f	t	d

The 8 ordered assignments shown are a tree model for the part of Φ shown. There are many others.

Q-Resolution

Two operations:

- *propositional resolution* on an *existential* clashing literal (tautologous resolvents prohibited);
  - Let $C_1 = [e, \alpha]$ and $C_2 = [\overline{e}, \beta]$.
  - $\mathbf{res}_e(C_1, C_2) = \alpha \cup \beta$.

- *universal reduction* on a *tailing universal* literal.
  - If u is inner scope to all existentials in clause C, it may be deleted (locally assigned *false* in C only).
  - Let $C = [u, \gamma]$.
  - $\mathbf{unrd}_u(C) = \gamma$.

Theorem [Kleine Büning,Karpinski,Flögel 1995]:

A PCNF Φ is *false* if and only if the empty clause is derivable by Q-resolution.

Remark (speaker's opinion):

Universal reduction is a major factor in recent success of practical QBF solvers.

QU-Resolution

A *QU-derivation* is the same as a Q-resolution derivation except that it includes resolutions in which the clashing literal is universal.

- Resolutions are still required to be non-tautologous.
- A QU-refutation is a QU-derivation of the empty clause.
- ***Regular*** QU-derivation: no variable appears twice as a clashing literal or reduction literal on any directed path through the derivation DAG.
- ***Ordered*** QU-derivation: variables appear in the same order on every directed path through the derivation DAG.
- ***Prefix-ordered*** QU-derivation: an ordered QU-derivation such that variables appear in outer to inner order of the quantifier prefix on paths directed away from the root of the derivation DAG.

Properties of QU-Resolution (CP 2012, to appear)

Lemma:
If clause D is derived from $\Psi = \overrightarrow{Q}\,.\,\mathcal{F}$ by QU-Resolution,
then D is logically implied by Ψ (i.e., tree models are preserved).

Theorem:
If (non-tautological) clause D is logically implied by $\Psi = \overrightarrow{Q}\,.\,\mathcal{F}$,
then $D^{(-)}$ can be derived from Ψ by prefix-ordered QU-Resolution.
If D is the empty clause then the QU-resolution can also be a Q-resolution.

QU-Resolution May Be Exponentially Shorter than Q-Resolution

The following QBF family is given by Kleine Büning and Lettman (1999) in the proof of their Theorem 7.4.8. The k-th QBF is:

$$\exists d_0 d_1 e_1 \forall x_1 \exists d_2 e_2 \forall x_2 \quad \cdots \quad \exists d_k e_k \forall x_k \exists f_1 \cdots \exists f_k .$$

$$\begin{array}{rl}
[\overline{d_0}] & \\
[d_0, \overline{d_1}, \overline{e_1}] & \\
[d_j, \overline{x_j}, \overline{d_{j+1}}, \overline{d_{j+1}}] & \quad \text{for } 1 \leq j < k \\
[e_j, x_j, \overline{d_{j+1}}, \overline{d_{j+1}}] & \quad \text{for } 1 \leq j < k \\
[d_k, \overline{x_k}, \overline{f_1}, \ldots, \overline{f_k}] & \\
[e_k, x_k, \overline{f_1}, \ldots, \overline{f_k}] & \\
[\overline{x_j}, f_j] & \quad \text{for } 1 \leq j \leq k \\
[x_j, f_j] & \quad \text{for } 1 \leq j \leq k
\end{array}$$

The proof of the theorem states that every Q-refutation of this formula has at least 2^k steps.

With QU-resolution, The overall number of steps is linear.

QU-Resolution and QBF ABstractions

Lonsing and Biere introduced *abstractions* of QBF for preprocessing purposes in SAT 2011.

The idea is developed further in SAT 2012 by Van Gelder, Wood, Lonsing.

Essentially the “abstraction” with respect to a variable p treats
all universal variables outer to p as though they were existential.
For inferential purposes, this amounts to using QU-resolution on these variables.

Existential Pure Literals

These are *not* logically implied from the assumptions.

So, treat as a new assumption.

An existential pure literal cannot have a quadrangle dependency on any universal literal, so it can move scopes without changing the truth value of the formula.

Universal Pure Literals

- These are *not* logically implied from the assumptions.
- So, treat as universal reductions (i.e., clause by clause).

Justification: No existential literal can have a quadrangle dependency on any universal pure literal, so the universal pure literal can "sink" to innermost scope without changing the truth value of the formula.

Clause Learning after Assuming a Pure Existential Literal

This example shows how a learned clause containing the negation of a (now) pure literal based on the original formula can occur and be useful.

The non-obvious part is that, although the partial assignment satisfies all *original* clauses that contain the negation of the now-pure literal, it does not satisfy the learned clause. (Not all clauses are shown.)

Φ	$\exists a_1$	$\forall u_2$	$\exists b_3$	$\forall u_4$	$\exists c_5$	$\exists a_5$	$\forall u_6$	$\exists c_7$	$\exists d_7$	$\exists e_7$	$\exists a_9$	$\exists b_9$	$\exists e_9$
C_1					c_5								e_9
C_2					c_5	$\overline{a_5}$							
C_3								c_7	d_7				
C_4								c_7	$\overline{d_7}$	e_7			
C_5									$\overline{d_7}$	$\overline{e_7}$			$\overline{e_9}$
C_6								$\overline{c_7}$	$\overline{d_7}$				
C_7								c_7	d_7	$\overline{e_7}$			
C_8								$\overline{c_7}$			a_9		
C_9								$\overline{c_7}$			$\overline{a_9}$		
C_{10}					$\overline{c_5}$	a_5						b_9	
C_{11}										$\overline{e_7}$		$\overline{b_9}$	

Learn $D_1 = [c_7, \overline{e_9}]$, using C_5, C_4, C_3.

Learn $D_2 = [c_5]$, using C_9, C_8, D_1, C_1.

With our proposal, learn $D_3 = [\overline{e_9}]$, using C_6, C_3, D_1.

Conclusion

QU-resolution provides a means to derive more clauses than Q-resolution.

Simply assume pure literals; don’t insist they are true (or false).

Better understanding of QBF theory.