A DPLL Algorithm for Solving DQBF

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What is DQBF?

- DQBF = Dependency Quantified Boolean Formulas
- Boolean formulas with Henkin quantifiers, i.e. dependencies are ...
  - ... specified explicitly
  - ... partially ordered
- Example: $\forall u_1, u_2 \exists e_1(u_1), e_2(u_2).\phi$
- Deciding DQBF is NEXPTIME-complete

Motivation

- Algorithm for solving NEXPTIME-problems (e.g. satisfiability of formulas in EPR or SMT: BV_UF)
- Profit from efficient techniques developed for SAT/QBF
- So far there is no algorithm for DQBF

Preliminaries

- DQBF $\psi = Q.\phi$, ϕ propositional matrix in CNF
- Q quantifier prefix of shape:
  $\forall u_1, \ldots, u_n \exists e_1(u_{1,1}, \ldots, u_{1,k_1}), \ldots, e_m(u_{m,1}, \ldots, u_{m,k_m})$,
  $u_{i,j} \in \{u_1, \ldots, u_n\}$
- $dep(e_i) := \{u_{i,1}, \ldots, u_{i,k_i}\}$ denotes the dependencies of e_i
- Consider assignments β_1, β_2. Formalization of dependence:
  $\forall u_j \in dep(e_i).\beta_1(u_j) = \beta_2(u_j) \rightarrow \beta_1(e_i) = \beta_2(e_i)$

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Pseudo-Code of DQDPLL

```
while (true) do
   state = CheckState(β);
   if (state == UNSAT) then
      HandleConflict();
   else if (state == SAT) then
      HandleSolution();
   else
      literal = SelectLiteral(β);
      AddDecision(β, literal);
   end if
end while
```

SelectLiteral

- A universal variable u_j can be picked at any time
- An existential variable e_i can be selected, if ...
$$\forall u_j \in dep(e_i).\beta(u_j) \neq ?$$
... i.e. all of its dependencies have already been assigned
- More freedom compared to QBF because branching on existential variables can be delayed

AddDecision

- The decision is saved on a decision stack
- For an existenial variable e_i, $dep(e_i) = \{u_{i,1}, \ldots, u_{i,k_i}\}$, a Skolem clause C_{sk} is created and linked to the decision
- For each decision on the stack the corresponding Skolem clause C_{sk} is considered to be part of the matrix and $F := \phi \wedge \bigwedge C_{sk}$ has to be satisfied
- $C_{sk} = (\overline{l(u_{i,1})} \wedge \cdots \wedge \overline{l(u_{i,k_i})} \wedge l(e_i))$ with $l(x) = \begin{cases} x, & \text{if } \beta(x) = 1 \\ \overline{x}, & \text{if } \beta(x) = 0 \end{cases}$
- This forces the algorithm to respect the dependencies

HandleSolution

- Look for the last universal that has been picked but the second branch has not been considered yet
- Restore the assignment at the point the universal variable was assigned
- No change to the decision stack occurs during this search

HandleConflict

- Look for the last existential that has been picked but the second branch has not been considered yet
- Backtrack and restore the assignment in the same way as it is done in QBF
- All touched decisions are removed from the stack during this search
- However backtracking takes place over trees/several branches because the decision stack was not touched in the SAT-case

A Simple Example

$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).\phi$$

$$\phi = (u_1 \oplus e_2) \land (u_2 \oplus e_1)$$

A Simple Example

$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).\phi$$

$$\begin{aligned} \phi &= (u_1 \oplus e_2) \wedge (u_2 \oplus e_1) \\ &= (u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1) \end{aligned}$$

A Simple Example

$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$$
$$F = \phi$$
$$\beta = (u_1 = ?, u_2 = ?, e_1 = ?, e_2 = ?)$$
$$F(\beta) = (u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$$

Stack:

A Simple Example

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Stack:

SelectLiteral: choose from $\{u_1, u_2\} \rightarrow u_1 = 0$
AddDecision: $(u_1 = 0, \text{LB}, \text{null})$

A Simple Example

$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$
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$F(\beta) = (e_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$

Stack:

$(u_1 = 0, \text{LB}, \text{null})$

SelectLiteral: choose from $\{u_2, e_1\} \rightarrow u_2 = 0$
AddDecision: $(u_2 = 0, \text{LB}, \text{null})$

A Simple Example

$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$$
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$$\beta = (u_1 = 0, u_2 = 0, e_1 = ?, e_2 = ?)$$
$$F(\beta) = (e_2) \wedge (e_1)$$

Stack:

$(u_2 = 0, \text{LB}, \text{null})$
$(u_1 = 0, \text{LB}, \text{null})$

A Simple Example

$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$$
$$F = \phi$$
$$\beta = (u_1 = 0, u_2 = 0, e_1 = ?, e_2 = ?)$$
$$F(\beta) = (e_2) \wedge (e_1)$$

Stack:

$(u_2 = 0, \mathrm{LB}, \mathrm{null})$
$(u_1 = 0, \mathrm{LB}, \mathrm{null})$

SelectLiteral: choose from $\{e_1, e_2\} \rightarrow e_1 = 1$
AddDecision: $(e_1 = 1, \mathrm{LB}, (u_1 \vee e_1))$

A Simple Example

$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$
$F = \phi \wedge (u_1 \vee e_1)$
$\beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = ?)$
$F(\beta) = (e_2)$

Stack:

$(e_1 = 1, \text{LB}, (u_1 \vee e_1))$
$(u_2 = 0, \text{LB}, \text{null})$
$(u_1 = 0, \text{LB}, \text{null})$

A Simple Example

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$$F(\beta) = (e_2)$$

Stack:

$(e_1 = 1, \mathrm{LB}, (u_1 \vee e_1))$
$(u_2 = 0, \mathrm{LB}, \mathrm{null})$
$(u_1 = 0, \mathrm{LB}, \mathrm{null})$

SelectLiteral: choose from $\{e_2\} \rightarrow e_2 = 1$
AddDecision: $(e_2 = 1, \mathrm{LB}, (u_2 \vee e_2))$

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$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$$
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$$\beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = 1)$$
$$F(\beta) = 1$$

Stack:

$(e_2 = 1, \text{LB}, (u_2 \vee e_2))$
$(e_1 = 1, \text{LB}, (u_1 \vee e_1))$
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$(u_1 = 0, \text{LB}, \text{null})$

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$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$
$F = \phi \wedge (u_1 \vee e_1) \wedge (u_2 \vee e_2)$
$\beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = 1)$
$F(\beta) = 1$

Stack:

$(e_2 = 1, \text{LB}, (u_2 \vee e_2))$
$(e_1 = 1, \text{LB}, (u_1 \vee e_1))$
$(u_2 = 0, \text{LB}, \text{null})$
$(u_1 = 0, \text{LB}, \text{null})$

HandleSolution: find latest universal LB decision
RestoreAssignment: $\beta = (u_1 = 0, u_2 = ?, e_1 = ?, e_2 = ?)$
AddDecision: $(u_2 = 1, \text{RB}, \text{null})$

A Simple Example

$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$$
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$$\beta = (u_1 = 0, u_2 = 1, e_1 = ?, e_2 = ?)$$
$$F(\beta) = (e_2) \wedge (\overline{e}_1) \wedge (e_1)$$

Stack:

$(u_2 = 1, \text{RB}, \text{null})$
$(e_2 = 1, \text{LB}, (u_2 \vee e_2))$
$(e_1 = 1, \text{LB}, (u_1 \vee e_1))$
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Stack:

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$(e_2 = 1, \text{LB}, (u_2 \vee e_2))$
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$F = \phi \wedge (u_1 \vee e_1) \wedge (u_2 \vee e_2) \wedge (u_1 \vee e_1)$

$\beta = (u_1 = 0, u_2 = 1, e_1 = 1, e_2 = ?)$

$F(\beta) = 0$

Stack:

$(e_1 = 1, \text{LB}, (u_1 \vee e_1))$
$(u_2 = 1, \text{RB}, \text{null})$
$(e_2 = 1, \text{LB}, (u_2 \vee e_2))$
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Stack:
$(e_1 = 1, \text{LB}, (u_1 \vee e_1))$
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$(e_2 = 1, \text{LB}, (u_2 \vee e_2))$
$(e_1 = 1, \text{LB}, (u_1 \vee e_1))$
$(u_2 = 0, \text{LB}, \text{null})$
$(u_1 = 0, \text{LB}, \text{null})$

HandleConflict: backtrack to latest existential LB decision
RestoreAssignment: $\beta = (u_1 = 0, u_2 = 1, e_1 = ?, e_2 = ?)$
AddDecision: $(e_1 = 0, \text{RB}, (u_1 \vee \overline{e}_1))$

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$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \vee e_2) \wedge (\overline{u}_1 \vee \overline{e}_2) \wedge (u_2 \vee e_1) \wedge (\overline{u}_2 \vee \overline{e}_1)$$
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Stack:
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HandleConflict: backtrack to latest existential LB decision

UNSAT

Concepts/Techniques adapted from SAT/QBF

- Unit Propagation
- Pure Literal Reduction
- Clause Learning
- Cube Learning
- Universal Reduction
- Watched Literal Schemes
- Selection Heuristics

Experiments

- Conversion of EPR formulas from the TPTP library to DQBF
- Comparison with a QBF solver on QBF benchmarks
- Generation of random DQBF instances

Conclusion and Future Work

- First DQBF solver
- DQDPLL architecture based on Skolem clauses
  + Consider expansion based solvers
- Translation for techniques from SAT/QBF
  + Measure single improvements
- Mixed results
  + Optimize and construct more natural benchmarks for DQBF

Questions?