Towards Efficient SAT Solving using XOR-OR-AND Normal Forms

work-in-progress

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joint work with B. Andraschko and M. Kreuzer

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XOR-OR-AND Normal Form

XNF

$$\bigwedge \begin{array}{l} (\neg X_1 \oplus X_3) \vee X_2 \\ (X_1 \oplus X_2 \oplus X_3) \vee \neg X_3 \\ X_3 \vee (X_1 \oplus X_2) \\ (\neg X_1 \oplus X_2) \vee X_1 \vee X_2 \end{array}$$

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XNF clause

XOR-OR-AND Normal Form

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Lineral: $(\neg X_1 \oplus X_3)$

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Proposition Every formula is equisatisfiable to a formula in 2-XNF.

$$Y \leftrightarrow (L_1 \vee L_2) \quad \equiv \quad (Y \vee \neg L_2) \wedge ((\neg Y \oplus L_1) \vee L_2).$$

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SCCs, Failed Linerals, ...

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implication graph: SCCs, Failed Linerals, ...

Equivalent XNF Clauses

Logic		Algebra
$X_1 \oplus X_2$	$\longleftrightarrow$	$x_1 + x_2 + 1$
$(X_1 \oplus X_2) \vee (\neg X_3 \oplus X_4)$		$(x_1 + x_2 + 1) \cdot (x_3 + x_4)$

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Corollary

$$C \text{ is a tautology} \quad \Longleftrightarrow \quad 1 \in V_C$$

`ANF_to_XNF` converts Algebraic Normal Form (ANF) to XNF.

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ASCON-128	format	#vars	#cls/polys	avg cls len
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$\longrightarrow$ cryptographic instances have *compact* representation

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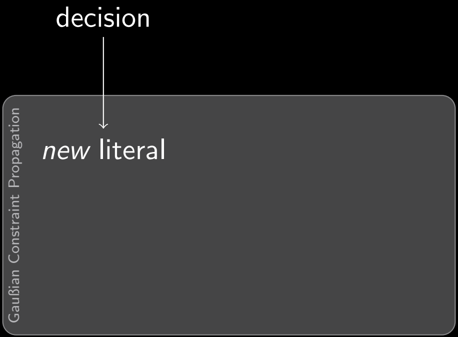

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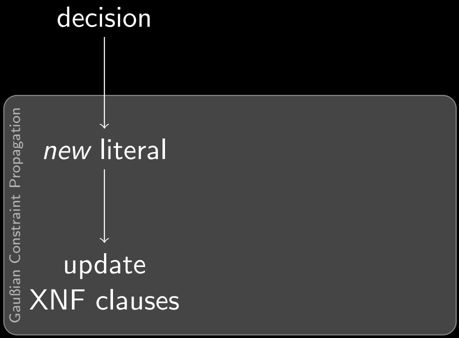

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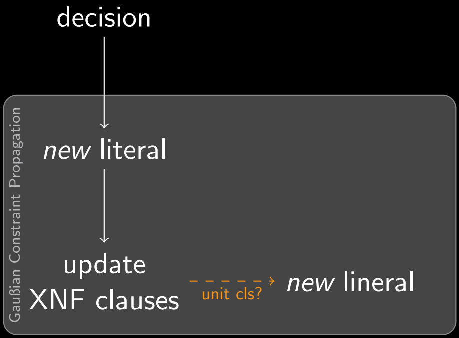

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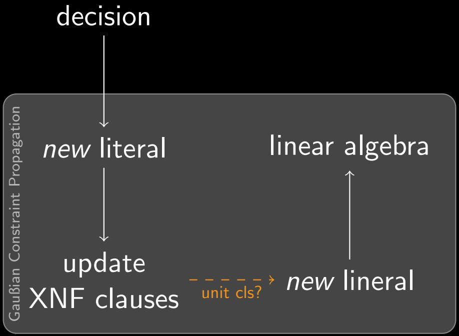

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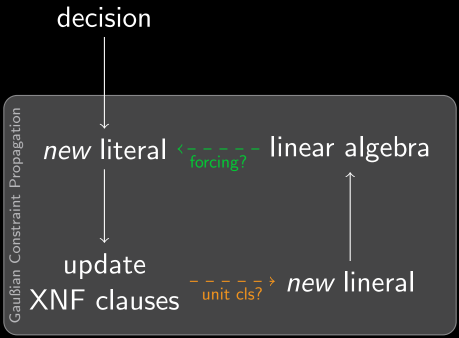

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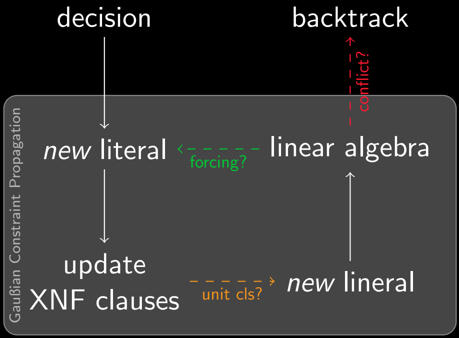

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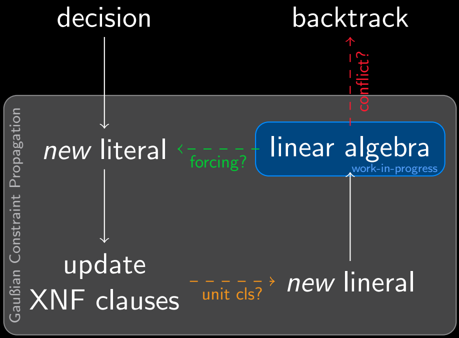

Watched Linerals

Problem watching distinct literals from distinct linerals

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$$\wr$$

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might miss unit clauses!

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Solution watch unshared literals from two distinct linerals

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$\longrightarrow$ XNF clauses can be efficiently managed by watch-lists

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s-resolution [Horacek]

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weaken clauses & change representation before resolution

$\rightsquigarrow$ expensive linear algebra?

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→ DPLL solver in C++ ✓

Experiments

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`XNF_to_CNF-XOR`

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$$\updownarrow$$

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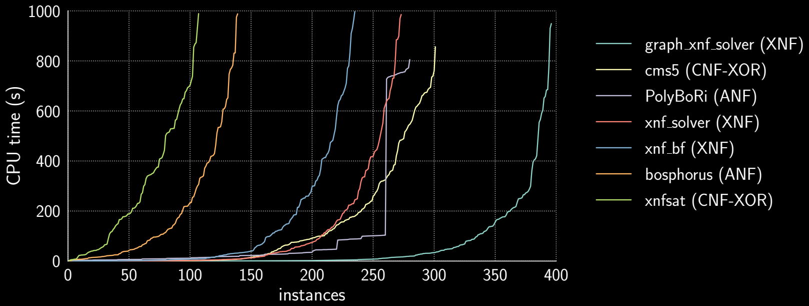

Figure: Cactus plots for 400 random *satisfiable* 2-XNF in n variables and $3n$ clauses where $n \in \{21, \dots, 40\}$.

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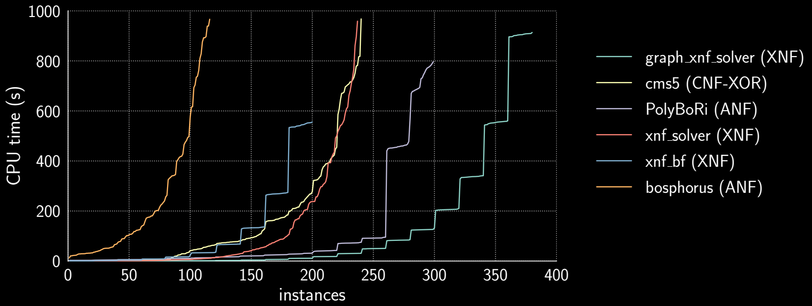

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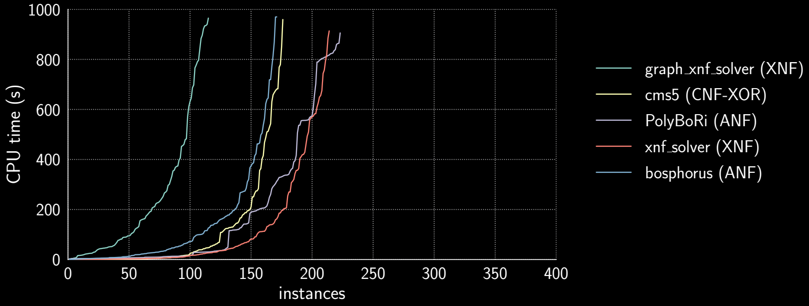

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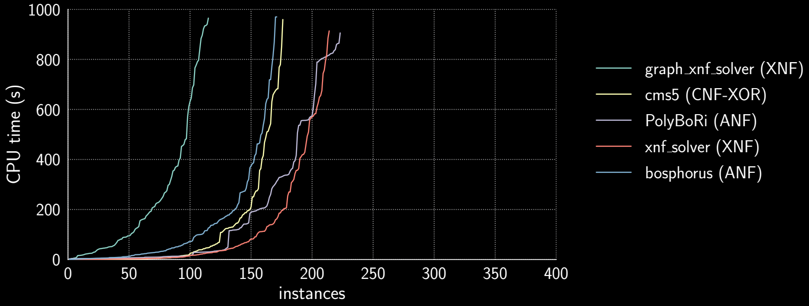

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Thank you for your attention!